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Thermal conduction of a circular inclusion with variable interface parameter

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Abstract

An imperfect bonding problem associated with a solitary circular inclusion embedded in an infinite matrix under a remotely applied uniform intensity is considered. Specifically, we study the effect of imperfect interfaces which are either of weakly or of highly conducting type and that the interface parameter could *vary arbitrarily* along the interface. By using the orthogonality properties of the trigonometric series, we show that the solution field is governed by a linear set of algebraic equations with an infinite number of unknowns. The governing matrix for the unknowns is primarily composed of elements which are simple combinations of the Fourier coefficients of the interface parameter. Solutions of the boundary-value problem are employed to estimate the effective conductivity tensor of a composite consisting of dispersions of circular inclusions with equal size. The effective properties solely depend on two particular constants among an infinite number of unknowns. It is demonstrated that, even for a composite with isotropic dispersions of inclusions, the composite may become effectively anisotropic due to the presence of a variable interface parameter. Further, we present two microstructure independent properties regarding the effective conductivity of the considered system. We first show that the effective conductivity tensor for a composite with variably imperfect interfaces is always diagonally symmetric. This is accomplished by means of a reciprocal relation that is established in such systems. Next, we present dual relations for the effective conductivities of two-dimensional composites with variably imperfect interfaces. The latter result is a direct consequence of the existence of a dual relation for the local fields in such composites, as pointed out by Benveniste and Miloh (Benveniste, Y., Miloh, T., 1999. J. Mech. Phys. Solids 47, 1873–1892).

Keywords: Imperfect interface; Effective conductivity; Variable interface parameter

1. Introduction

This work is concerned with the thermal conduction of a composite with imperfect interfaces. The content of this work can be divided into three parts: (1) to resolve the field quantities of the boundary-value problem associated with a circular inclusion imperfectly embedded in an infinite matrix with *arbitrarily varying interface parameter*, (2) to assess the macroscopic behavior of the composite incorporating the effect

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of a variable interface parameter and (3) to present two microstructure independent properties of composites with variable interface parameter.

There are two types of imperfect interfaces in the context of thermal conduction, which can be modeled as a thin interface layer consisting of either weakly conducting or highly conducting phase. At weakly conducting interphase, the temperature potential jumps across the interface. The associated normal component of the heat flux is continuous and is proportional to the jump in temperature potential (Sanchez-Palencia, 1970). The effect of the interface of this kind has been investigated by a number of researchers (see for example, Benveniste and Miloh (1986), Cheng and Torquato (1997) and the references cited therein). On the other hand, at highly conducting interfaces, the temperature is continuous across the interface, whereas the normal component of the heat flux has a discontinuity which is proportional to a certain differential expression of the temperature (Pham Huy and Sanchez-Palencia, 1974). Studies of the effect of the interface of this kind appear to be relatively recent (Miloh and Benveniste, 1999). The proportional constant of imperfect interfaces, $\beta(\theta)$ or $\alpha(\theta)$, is referred to as an interface parameter that may not be spatially uniform along the interface.

In this work, we examine the solution of a boundary-value problem associated with a circular inclusion embedded in an infinite matrix under a remotely applied uniform intensity along a certain direction. We examine the effect of imperfect interface in which the interface could be either of *weakly* or of *highly conducting type* and that the interface parameter could *vary arbitrarily* along the interface. By using the orthogonality properties of the trigonometric series, we show that the solution is governed by a linear set of algebraic equations. The system can be constructed upto any desired order N, and the solutions are resolved by an appropriate truncation. The elements of the governing matrix are explicitly expressed, which are only simple combinations of the Fourier coefficients of the interface parameter together with the phase properties. For an interface parameter that is capable of being expressed by a *finite* cosine series, the governing matrix becomes a banded one. A principal feature of the present formulation is that the solution procedure is straightforward and mathematically simple. For a different variation of β or α, no further derivation is needed and the governing system for the solutions remains unchanged simply by changing the new set of Fourier coefficients of α or β.

In the literature, various aspects of the effects of imperfect interfaces have been examined, including micromechanical estimates of the effective properties, variational bounds, exact connections of microstructure independent relationships, etc. To the author's knowledge, except the recent work by Ru and Schiavone (1997), [1] most of the studies on imperfect bonding problems exclusively dealt with *constant* interface parameter. Particularly, Ru and Schiavone examined a circular inclusion in an infinite medium under an antiplane shear, which is mathematically equivalent to the present one. However, *only* weakly conducting type of interface (in the terminology of conduction) is examined in their work. They employed a complex variable approach together with the analytic continuation method to analyze the solutions. In contrast to the present formulation, their derivations are mathematically cumbersome. Particularly, it may present some mathematical difficulties for a complicated function of $\beta(\theta)$ or $\alpha(\theta)$. Our solution is simple and explicit which is basically governed by a linear set of algebraic equations.

The obtained results are employed to estimate the effective conductivities of a composite consisting of circular cylinders of equal size under the dilute approximation and the Mori–Tanaka (1973) mean field approximation. It is seen that even for an isotropic arrangement of the circular inclusions, the effective conductivity tensor of the medium may become macroscopically anisotropic due to the effect of a variable interface parameter.

Lastly, we present two microstructure independent properties regarding the composite system with variable interface parameters. The first one is on the diagonal symmetry of the effective conductivity tensor.

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[1] The Author would like to thank Y. Benveniste for bringing the paper to his attention during the writing stage.

The fact that the effective conductivity tensor in composites with perfect interfaces is diagonally symmetric is well known. One may wonder whether this property prevails in the case of variably imperfect interfaces as well. The answer is yes. The proof is achieved by extending first the well-known reciprocal theorem to the case of imperfect interfaces. The second result is on a dual relation between the effective conductivity tensor of composites with variable interface parameters. The duality relations have their origins in the works of Keller (1964), Dykhne (1971) and Mendelson (1975). Later developments in the same setting and in more general heterogeneous media have been derived by many researchers (Helsing et al., 1997; Milton, 1997). Recently, Benveniste and Miloh (1999) showed that the dual relation exists between *the local fields of* a composite system with variable interface parameter. Benveniste (1999) also pointed out that the duality principle can in fact be applied to anisotropic constituents. In Proposition 2, we present the dual relation on the effective conductivity tensor for such systems. This result is certainly anticipated from the works of Benveniste (1999) and Benveniste and Miloh (1999); nevertheless, they have not been explicitly stated before.

The outline of the paper is as follows: A description of the interface conditions in both kinds of interfaces is given in Section 2. Section 3 resolves the considered boundary-value problems with imperfect interface of both types. Some particular examples of variations of $\alpha(\theta)$ and $\beta(\theta)$ are presented in Section 4. Section 5 examines the effect of imperfect interface on the effective conductivity of composites. Section 6 presents two microstructure independent properties of the considered system. We finally mention that the mathematical frameworks of heat conduction, electric conduction, dielectric behavior, magnetic permeability and anti-plane deformation in cylindrical aggregates are entirely equivalent. Any results obtained in one area can be readily applied to the other domains.

2. Preliminary

We consider a two-phase medium in which the conductivity of each phase is denoted by k_i, where $i = 1, 2$. Each of the phases occupies region $V_i,\ i = 1, 2$, that are separated by the interface Γ. Let T be the temperature field, the intensity H_i and heat flux q_i are, respectively, given by $H_i = -\nabla T$, $q_i = kH_i$. Under steady state conditions, the heat flux is divergenceless and thus in $V\ (= V_1 + V_2)$ obeys

$$\nabla^2 T = 0 \quad \text{in } V. \tag{1}$$

For a weakly conducting interface, the normal component of the heat flux is continuous, whereas the temperature field undergoes a discontinuity which is proportional to the normal component of the heat flux:

$$k_1 \frac{\partial T_1}{\partial n} = k_2 \frac{\partial T_2}{\partial n} = \beta(T_1 - T_2)\bigg|_{\Gamma}. \tag{2}$$

Here, $\partial/\partial n$ is the normal derivative on Γ from phase 2 to phase 1 and $\beta(\theta)$ ($\geqslant 0$), referred to as the interface parameter in the sequel, is defined by

$$\beta = \lim_{\substack{t\to 0\\ k_c \to 0}} k_c/t, \tag{3}$$

where k_c and t denote respectively the interphase conductivity and its thickness and is always not less than zero. In particular, the scalar parameter β could vary along the interface Γ. Eq. (2) indicates that there exists a thermal resistance at the interface between two phases. In the case where the interface is perfectly bonded, the interface parameter $\beta \to \infty$, whereas $\beta = 0$ stands for adiabatic contact.

For a highly conducting interface, the temperature field is continuous, whereas the normal component of the heat flux undergoes a discontinuity:

$$T_1 = T_2|_\Gamma, \qquad k_2 \frac{\partial T_2}{\partial n} - k_1 \frac{\partial T_1}{\partial n} = \alpha \Delta_s T + \nabla_s T \cdot \nabla_s \alpha \bigg|_\Gamma . \tag{4a,b}$$

Here, $\alpha(\theta)$ ($\geqslant 0$) is the scalar interface parameter defined as

$$\alpha = \lim_{\substack{t\to 0\\ k_c \to 0}} k_c t, \tag{5}$$

where Δ_s is the operator of surface Laplacian and ∇_s is the operator of surface gradient (Van Bladel, 1964). These operators can be written in terms of two orthogonal parametric curvilinear coordinates (u_1, u_2) that describe the interface

$$\Delta_s = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1}{h_2}\frac{\partial}{\partial u_2}\right)\right], \qquad \nabla_s = \frac{\mathbf{u}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{u}_2}{h_2}\frac{\partial}{\partial u_2}, \tag{6}$$

where h_i is the metric coefficient and $\mathbf{u}_i$ denotes the unit vector of these curvilinear coordinates. For a circular boundary, it follows simply that

$$\nabla_s = \frac{\mathbf{u}_\theta}{r}\frac{\partial}{\partial \theta}, \qquad \Delta_s = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{7}$$

in a cylindrical coordinate (r, θ, z).

It is seen that if the interface parameter α is a constant, Eq. (4a,b) will take a simpler form since the second term on the right-hand side of Eq. (4a,b) vanishes. In the case where the interface is perfectly bonded, the interface parameter $\alpha \to 0$, whereas a value of $\alpha \to \infty$ describes contact with a medium of infinite conductivity.

3. A circular inclusion in an infinite matrix

We consider the boundary-value problem of a circular cylinder of radius a (with conductivity k_2) in an unbounded matrix of conductivity k_1 subjected to a uniform intensity field H_i at the remote boundary:

$$T(x, y)|_{r\to\infty} = -H_i x_i. \tag{8}$$

3.1. Weakly conducting interface with variable interface parameter β

We first consider the case of a weakly conducting interface (2). Without loss of generality, we assume that the interface parameter $\beta(\theta)$ is at least piecewise continuous and integrable in an interval $[0, 2\pi]$ so that one can write

$$\beta(\theta) = \frac{1}{2}\beta_0 + \sum_{n=1}^{\infty}\left(\beta_n' \cos n\theta + \beta_n'' \sin n\theta\right) \geqslant 0. \tag{9}$$

Let us first consider that a uniform intensity is applied in the x_1 direction. For isotropic phases, the temperature field fulfills the Laplace equation (1). Thus, the potential inside the cylinder is given by

$$T_2(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta), \tag{10}$$

and the potential outside the cylinder by

$$T_1(r,\theta) = C_0 + \sum_{n=1}^{\infty} [r^{-n}(C_n \cos n\theta + D_n \sin n\theta) + r^n (E_n \cos n\theta + F_n \sin n\theta)]. \tag{11}$$

The remote boundary condition readily implies that $C_0 = 0$, $F_n = 0$, $E_1 = -H_1$, and $E_n = 0$ for $n \neq 1$. In addition, the continuity of the normal component of heat flux provides that

$$C_1 = -a^2(kA_1 + H_1), \qquad C_n = -a^{2n}kA_n, \quad n = 2,3,\ldots,$$

$$D_n = -a^{2n}kB_n, \quad n = 1,2,\ldots, \tag{12}$$

where $k = k_2/k_1$. To satisfy the second equality of Eq. (2), namely

$$-k_2\left[\sum_{n=1}^{\infty} na^{n-1}(A_n \cos n\theta + B_n \sin n\theta)\right] = \beta(\theta)\left[A_0 + 2H_1 a\cos\theta + \sum_{n=1}^{\infty} a^n(1+k)(A_n \cos n\theta + B_n \sin n\theta)\right], \tag{13}$$

we need further derivations. The right-hand side of Eq. (13) involves a product of two infinite trigonometric series. To proceed, we employ the known identity of products of Fourier series (Tolstov, 1976, pp. 124–125). Particularly, given two periodic functions $\lambda(\theta)$ and $\beta(\theta)$ defined on the interval $[0, 2\pi]$, suppose the product of these two functions is defined by a new function $\gamma(\theta)(=\lambda(\theta)\beta(\theta))$. Then, the Fourier coefficients of λ, β and γ are connected by Eqs. (6.3) and (6.4) of Tolstov (1976).

For convenience, we rewrite the results as a set of infinite equations

$$\begin{bmatrix} \beta_0/2 & (\mathbf{b}')^{\mathrm{t}} & (\mathbf{b}'')^{\mathrm{t}} \\ \mathbf{b}' & \mathbf{B}_\beta & \mathbf{C}_\beta \\ \mathbf{b}'' & -\mathbf{C}_\beta & \mathbf{B}_\beta \end{bmatrix} \begin{Bmatrix} \lambda_0 \\ \mathbf{a}' \\ \mathbf{a}'' \end{Bmatrix} = \begin{Bmatrix} \gamma_0 \\ 2\mathbf{r}' \\ 2\mathbf{r}'' \end{Bmatrix}, \tag{14}$$

where

$$\begin{aligned} (\mathbf{a}')^{\mathrm{t}} &= \left[\lambda'_1, \lambda'_2, \ldots, \lambda'_N, \ldots\right], & (\mathbf{a}'')^{\mathrm{t}} &= \left[\lambda''_1, \lambda''_2, \ldots, \lambda''_N, \ldots\right], \\ (\mathbf{b}')^{\mathrm{t}} &= \left[\beta'_1, \beta'_2, \ldots, \beta'_N, \ldots\right], & (\mathbf{b}'')^{\mathrm{t}} &= \left[\beta''_1, \beta''_2, \ldots, \beta''_N, \ldots\right], \\ (\mathbf{r}')^{\mathrm{t}} &= \left[\gamma'_1, \gamma'_2, \ldots, \gamma'_N, \ldots\right], & (\mathbf{r}'')^{\mathrm{t}} &= \left[\gamma''_1, \gamma''_2, \ldots, \gamma''_N, \ldots\right], \end{aligned} \tag{15}$$

superscript t being the matrix transpose, and

$$\mathbf{B}_\beta = \begin{bmatrix} \beta_0 + \beta'_2 & \beta'_1 + \beta'_3 & \beta'_2 + \beta'_4 & \cdots & \beta'_{N-1} + \beta'_{N+1} & \\ & \beta_0 + \beta'_4 & \beta'_1 + \beta'_5 & \cdots & \beta'_{N-2} + \beta'_{N+2} & \\ & & \beta_0 + \beta'_6 & \cdots & \vdots & \\ \vdots & \vdots & & \ddots & & \\ & \text{sym} & & & \beta_0 + \beta'_{2N} & \\ & \vdots & & & \vdots & \ddots \end{bmatrix}, \tag{16}$$

$$\mathbf{C}_\beta = \begin{bmatrix} \beta_2'' & \beta_1''+\beta_3'' & \beta_2''+\beta_4'' & & \beta_{N-1}''+\beta_{N+1}'' & \\ \beta_3''-\beta_1'' & \beta_4'' & \beta_1''+\beta_5'' & \cdots & \beta_{N-2}''+\beta_{N+2}'' & \\ \beta_4''-\beta_2'' & \beta_5''-\beta_1'' & \beta_6'' & \cdots & \vdots & \\ \vdots & \vdots & & \ddots & & \\ \beta_{N+1}''-\beta_{N-1}'' & \beta_{N+2}''-\beta_{N-2}'' & \beta_{N+3}''-\beta_{N-3}'' & & \beta_{2N}'' & \\ & \vdots & & & \vdots & \ddots \end{bmatrix}. \tag{17}$$

It is seen that the matrix $\mathbf{B}_\beta$ is diagonally symmetric, whereas the matrix $\mathbf{C}_\beta$ is not. The identity in Eq. (14) could be of use in some applications. For example, the reciprocal of any trigonometric series can be easily resolved by letting $\gamma_0 = 2$, $\mathbf{r}' = \mathbf{r}'' = \mathbf{0}$. We have checked for a few particular examples using MAPLE V.

Back to Eq. (13), one may set $\lambda_0 = 2A_0$, $\lambda_n' = (1+k)A_n a^n$ for $n \geqslant 1$ and $\lambda_n'' = (1+k)B_n a^n$ for $n \geqslant 1$. In addition, we identify γ_n' with $-nk_2 a^{n-1}A_n$, and γ_n'' with $-nk_2 a^{n-1}B_n$ for $n \geqslant 1$ and $\gamma_0 = 0$. It then follows that

$$\mathbf{r}' = -\frac{k_\mathrm{h}}{2a}\Lambda\mathbf{a}', \qquad \mathbf{r}'' = -\frac{k_\mathrm{h}}{2a}\Lambda\mathbf{a}'', \tag{18}$$

where

$$\Lambda = \mathrm{diag}[1, 2, 3, \ldots, N, \ldots], \tag{19}$$

and k_h is the harmonic mean of phase conductivities given by

$$k_\mathrm{h} = 2\left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1}. \tag{20}$$

Note that the interface condition (13) (right-hand side) is now transformed into a linear set of equations with an infinite number of unknowns A_n and B_n. Upon rearrangements of the unknown and known quantities, we obtain a linear system for the unknown quantities A_n and B_n as

$$\begin{bmatrix} \beta_0/2 & (\mathbf{b}')^\mathrm{t} & (\mathbf{b}'')^\mathrm{t} \\ \mathbf{b}' & \mathbf{B}_\beta + \xi\Lambda & \mathbf{C}_\beta \\ \mathbf{b}'' & -\mathbf{C}_\beta & \mathbf{B}_\beta + \xi\Lambda \end{bmatrix} \begin{Bmatrix} 2A_0 \\ \mathbf{x} \\ \mathbf{y} \end{Bmatrix} = -2H_1 a \begin{Bmatrix} \beta_1' \\ \mathbf{p} \\ \mathbf{q} \end{Bmatrix}, \tag{21}$$

where $\xi = k_\mathrm{h}/a$ and

$$\begin{aligned} \mathbf{x}^\mathrm{t} &= (1+k)\left[A_1 a,\ A_2 a^2, \ldots,\ A_n a^n, \ldots\right], \\ \mathbf{y}^\mathrm{t} &= (1+k)\left[B_1 a,\ B_2 a^2, \ldots,\ B_n a^n, \ldots\right], \\ \mathbf{p}^\mathrm{t} &= \left[\left(\beta_0 + \beta_2'\right), \left(\beta_1' + \beta_3'\right), \ldots, \left(\beta_{N-1}' + \beta_{N+1}'\right), \ldots\right], \\ \mathbf{q}^\mathrm{t} &= \left[-\beta_2'', \left(\beta_1'' - \beta_3''\right), \ldots, \left(\beta_{N-1}'' - \beta_{N+1}''\right), \ldots\right]. \end{aligned} \tag{22}$$

System (22) can be solved by truncation in any desired order. We remark that as long as $\beta(\theta)$ is not uniform, system (21) will involve an infinite number of unknowns to be determined. The entire fields can now be resolved.

Exactly the same steps can be performed when a uniform intensity is applied in the x_2 direction, namely $T(x, y)|_{r\to\infty} = -H_2 x_2$. For convenience, we denote the unknowns by the same variables with a superscript $\perp$. The governing system is very similar to Eq. (21), with only a change on the right-hand side

$$\begin{bmatrix} \beta_0/2 & (\mathbf{b}')^{\mathrm{t}} & (\mathbf{b}'')^{\mathrm{t}} \\ \mathbf{b}' & \mathbf{B}_\beta + \xi\Lambda & \mathbf{C}_\beta \\ \mathbf{b}'' & -\mathbf{C}_\beta & \mathbf{B}_\beta + \xi\Lambda \end{bmatrix} \begin{Bmatrix} 2A_0^\perp \\ \mathbf{x}^\perp \\ \mathbf{y}^\perp \end{Bmatrix} = -2H_2 a \begin{Bmatrix} \beta_1'' \\ -\mathbf{q} \\ \mathbf{p} \end{Bmatrix}. \tag{23}$$

3.2. Highly conducting interface with variable interface parameter α

We consider the same boundary-value problem as in the previous subsection, but with a highly conducting interface. The interface parameter α allows to be a variable around the circular boundary

$$\alpha(\theta) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \left(\alpha_n' \cos n\theta + \alpha_n'' \sin n\theta\right) \geqslant 0. \tag{24}$$

The potential inside and outside the cylinder will take the same form as in Eqs. (10) and (11). The remote boundary condition in the x_1 direction implies that $C_0 = 0$, $F_n = 0$, $E_1 = -H_1$ and $E_n = 0$ for $n \neq 1$, and the continuity of the temperature at $r = a$ provides

$$C_1 = (A_1 + H_1)a^2, \qquad C_n = A_n a^{2n} \quad \text{for} \quad n = 2, 3, \ldots, \qquad D_n = B_n a^{2n} \quad \text{for} \quad n = 1, 2, \ldots \tag{25}$$

Applying Eqs. (24), (25), and (10) into Eq. (4b), we obtain the condition on the interface boundary $r = a$:

$$\begin{aligned} &2k_1\left\{2H_1\cos\theta + \sum_{n=1}^{\infty} n(k+1)a^{n-1}[A_n\cos n\theta + B_n \sin n\theta]\right\} \\ &\quad = -\alpha_0 \sum_{n=1}^{\infty} n^2 a^{n-2}(A_n \cos n\theta + B_n \sin n\theta) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{n-2}\left\{A_n\alpha_m'\left[\left(mn - n^2\right)\cos(m-n)\theta\right.\right. \\ &\qquad \left.\left. - \left(mn + n^2\right)\cos(m+n)\theta\right]\right\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{n-2}\left\{A_n\alpha_m''\left[\left(mn - n^2\right)\sin(m-n)\theta\right.\right. \\ &\qquad \left.\left. - \left(mn + n^2\right)\sin(m+n)\theta\right]\right\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{n-2}\left\{B_n\alpha_m'\left[\left(-mn + n^2\right)\sin(m-n)\theta\right.\right. \\ &\qquad \left.\left. - \left(mn + n^2\right)\sin(m+n)\theta\right]\right\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{n-2}\left\{B_n\alpha_m''\left[\left(mn - n^2\right)\cos(m-n)\theta\right.\right. \\ &\qquad \left.\left. + \left(mn + n^2\right)\cos(m+n)\theta\right]\right\}. \end{aligned} \tag{26}$$

To proceed, we multiply Eq. (26) by $\cos\theta$ and $\sin\theta$, and integrate from 0 to 2π. Using the orthogonality relations of the trigonometric series, after some manipulations, we find

$$\begin{aligned} &\left(\alpha_2' - \alpha_0\right)A_1 a^{-1} + \sum_{n=2}^{\infty} A_n n a^{n-2}\left(\alpha_{n+1}' - \alpha_{n-1}'\right) + \alpha_2'' B_1 a^{-1} + \sum_{n=2}^{\infty} B_n n a^{n-2}\left(\alpha_{n+1}' - \alpha_{n-1}'\right) \\ &\quad = 2k_1[2H_1 + (k+1)A_1], \end{aligned} \tag{27}$$

$$\alpha_2' A_1 a^{-1} + \sum_{n=2}^{\infty} A_n n a^{n-2}\left(\alpha_{n+1}' + \alpha_{n-1}'\right) - \left(\alpha_0 + \alpha_2'\right)B_1 a^{-1} - \sum_{n=2}^{\infty} B_n n a^{n-2}\left(\alpha_{n+1}' + \alpha_{n-1}'\right) = 2k_1(k+1)B_1. \tag{28}$$

In a similar manner, multiplying Eq. (26) by $\cos p\theta$ and $\sin p\theta$, $p \geqslant 2$, and integrating from 0 to 2π, will give

$$2k_1(k+1)pa^{p-1}A_p = \sum_{n=1}^{p-1} A_n n a^{n-2} p(\alpha'_{p+n} - \alpha'_{p-n}) + A_p p^2 a^{p-2}(\alpha'_{2p} - \alpha_0) + \sum_{n=p+1}^{\infty} A_n n a^{n-2} p(\alpha'_{n+p} - \alpha'_{n-p})$$
$$+ \sum_{n=1}^{p-1} B_n n a^{n-2} p(\alpha''_{p+n} + \alpha''_{p-n}) + B_p p^2 a^{p-2}\alpha''_{2p} + \sum_{n=p+1}^{\infty} B_n n a^{n-2} p(\alpha''_{n+p} - \alpha''_{n-p}), \tag{29}$$

$$2k_1(k+1)pa^{p-1}B_p = \sum_{n=1}^{p-1} A_n n a^{n-2} p(\alpha''_{p+n} - \alpha''_{p-n}) + A_p p^2 a^{p-2}\alpha''_{2p} + \sum_{n=p+1}^{\infty} A_n n a^{n-2} p(\alpha''_{n+p} + \alpha''_{n-p})$$
$$- \sum_{n=1}^{p-1} B_n n a^{n-2} p(\alpha'_{p+n} + \alpha'_{p-n}) - B_p p^2 a^{p-2}\alpha'_{2p} - \sum_{n=p+1}^{\infty} B_n n a^{n-2} p(\alpha'_{n+p} + \alpha'_{n-p}). \tag{30}$$

Note that, by the definition of Eq. (22), Eqs. (27–30) can be expanded as a set of infinite equations with unknowns x_n and y_n. In particular, the system can be written in the matrix form as

$$\left\{ \begin{bmatrix} \mathbf{A}_\alpha & \mathbf{C}_\alpha^{\mathrm{t}} \\ \mathbf{C}_\alpha & -\mathbf{B}_\alpha \end{bmatrix} - 4\eta \begin{bmatrix} \Lambda^{-1} & \mathbf{0} \\ \mathbf{0} & \Lambda^{-1} \end{bmatrix} \right\} \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{y} \end{Bmatrix} = 8\eta a H_1 \begin{Bmatrix} \mathbf{h} \\ \mathbf{0} \end{Bmatrix}, \tag{31}$$

where $\eta = ak_a$, $(\mathbf{h})^{\mathrm{t}} = [1, 0, 0, \ldots]$, and $k_a = (k_1 + k_2)/2$ denotes the arithmetic mean of phase conductivities. The matrices $\mathbf{B}_\alpha$ and $\mathbf{C}_\alpha$ are defined as in Eqs. (16) and (17) with the elements now being replaced by the Fourier coefficients of $\alpha(\theta)$, and the matrix $\mathbf{A}_\alpha$ is diagonally symmetric defined as

$$\mathbf{A}_\alpha = \begin{bmatrix} \alpha'_2 - \alpha_0 & \alpha'_3 - \alpha'_1 & \alpha'_4 - \alpha'_2 & \cdots & \alpha'_{N+1} - \alpha'_{N-1} & \cdots \\ & \alpha'_4 - \alpha_0 & \alpha'_5 - \alpha'_1 & \cdots & \alpha'_{N+2} - \alpha'_{N-2} & \\ & & \alpha'_6 - \alpha_0 & \cdots & \alpha'_{N+3} - \alpha'_{N-3} & \\ & & & \ddots \ \cdots & \vdots & \\ & \text{sym} & & \ddots & \alpha'_{2N-1} - \alpha'_1 & \\ & & & & \alpha'_{2N} - \alpha_0 & \\ & & & & & \ddots \end{bmatrix}. \tag{32}$$

To solve the system, again one may truncate the fields by any desired order N. We note that even if $\alpha(\theta)$ is only a finite Fourier cosine series with order N, namely, $\alpha_{N+1} = \alpha_{N+2} = \ldots = 0$, then system (31) still involves an infinite number of unknowns. However, in this case all the nonzero elements of $\mathbf{A}_\alpha$, $\mathbf{B}_\alpha$, $\mathbf{C}_\alpha$ are confined within a band with bandwidth $2N - 1$.

Suppose now a uniform intensity H_2 is prescribed at the remote boundary, the solution can be derived in a similar manner, and system (31) is now changed to

$$\left\{ \begin{bmatrix} \mathbf{A}_\alpha & \mathbf{C}_\alpha^{\mathrm{t}} \\ \mathbf{C}_\alpha & -\mathbf{B}_\alpha \end{bmatrix} - 4\eta \begin{bmatrix} \Lambda^{-1} & \mathbf{0} \\ \mathbf{0} & \Lambda^{-1} \end{bmatrix} \right\} \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda \end{bmatrix} \begin{Bmatrix} \mathbf{x}^\perp \\ \mathbf{y}^\perp \end{Bmatrix} = 8\eta a H_2 \begin{Bmatrix} \mathbf{0} \\ \mathbf{h} \end{Bmatrix}. \tag{33}$$

4. Some examples of $\beta(\theta)$ and $\alpha(\theta)$

We examine a few specific examples of $\beta(\theta)$ and $\alpha(\theta)$. First, it is noted that the coefficient β_0 in Eq. (9) should always be positive; $\beta_0 = 0$ occurs only when the interface parameter vanishes throughout the whole interface. This is simply due to the fact that $\beta_0/2$ is exactly the mean value of $\beta(\theta)$ around the circular interface. The same reasoning also applies to α_0.

Let us now first consider that the interface parameter β is a pure cosine series. In this case, $\mathbf{b}'' = \mathbf{0}$ and $\mathbf{C}_\beta = \mathbf{0}$. Thus, it is evident that $\mathbf{y} = \mathbf{0}$, which means that the coefficients B_ns are all zero. Particularly, for a finite cosine series, say $\beta(\theta) = \beta_0/2 + \beta_1' \cos\theta$, the system of Eq. (21) becomes

$$\begin{bmatrix} \beta_0/2 & \beta_1' & 0 & \cdots & & \cdots \\ & \beta_0 + \xi & \beta_1' & 0 & & \\ & & \ddots & \ddots & 0 & \ddots \\ & & & & & \ddots \\ & \text{sym} & & & \beta_0 + n\xi & \ddots \\ & & & & & \ddots \end{bmatrix} \begin{Bmatrix} 2A_0 \\ A_1 a(1+k) \\ A_2 a^2 (1+k) \\ \vdots \\ A_n a^n (1+k) \\ \vdots \end{Bmatrix} = -2Ha \begin{Bmatrix} \beta_1' \\ \beta_0 \\ \beta_1' \\ 0 \\ 0 \\ \vdots \end{Bmatrix}. \tag{34}$$

It is seen that the left-hand side of Eq. (34) is a symmetric "tridiagonal" matrix. In general, for a finite cosine series of N terms, the left-hand side of Eq. (34) is a symmetric matrix with bandwidth $N + 1$. The term *bandwidth* means that all nonzero coefficients of the matrix appear in a banded area surrounding the main diagonal terms. For the case of constant parameter, i.e. $\beta = \beta_0/2$, the matrix $\mathbf{B}_\beta$ reduces to a diagonal matrix

$$\mathbf{B}_\beta = \operatorname{diag}[\beta_0 + \xi,\ \beta_0 + 2\xi, \ldots,\ \beta_0 + N\xi, \ldots], \tag{35}$$

and the matrices $\mathbf{b}'$, $\mathbf{b}''$, $\mathbf{p}''$ and $\mathbf{C}_\beta$ vanish identically and $(\mathbf{p}')^{\mathrm{t}} = [\beta_0, 0, 0, \ldots]$. In this case, system (21) can be exactly resolved as

$$A_1 = -\frac{2Ha\beta_0}{a\beta_0(1+k) + 2k_2}, \qquad A_n = 0 \quad \text{for} \quad n \neq 1, \ \ B_n = 0. \tag{36}$$

We have verified that the field solutions of the boundary-value problem recover the known results for a constant interface parameter (Benveniste, 1987a).

On the other hand, if all $\beta_n' = 0$, $n \neq 0$, then system (21) becomes

$$\begin{bmatrix} \beta_0/2 & \mathbf{0} & (\mathbf{b}'')^{\mathrm{t}} \\ \mathbf{0} & \xi\Lambda & \mathbf{C}_\beta \\ \mathbf{b}'' & -\mathbf{C}_\beta & \xi\Lambda \end{bmatrix} \begin{Bmatrix} 2A_0 \\ \mathbf{x} \\ \mathbf{y} \end{Bmatrix} = -2Ha \begin{Bmatrix} 0 \\ \mathbf{0} \\ \mathbf{p}'' \end{Bmatrix} \tag{37}$$

which indicates that the coefficients A_0, A_n and B_n are in general nonzero.

For a highly conducting interface, we first consider that the interfacial parameter α is a cosine series. In this case, the matrix $\mathbf{C}_\alpha = \mathbf{0}$ and thus $\mathbf{y} = \mathbf{0}$, which means that the coefficients B_ns are all zero. Particularly, for a finite cosine series, say $\alpha(\theta) = \alpha_0/2 + \alpha_1' \cos\theta$, system (31) becomes

$$\begin{bmatrix} \alpha_0 - 4\eta & \alpha_1' & 0 & \cdots & \\ & \alpha_0 - 2\eta & \alpha_1' & 0 & \\ & & \ddots & \alpha_1' & \ddots \\ & \text{sym} & & \alpha_0 - 4\eta/n & \ddots \\ & & & & \ddots \end{bmatrix} \begin{Bmatrix} A_1 a(1+k) \\ A_2 a^2 (1+k)/2 \\ \vdots \\ A_n a^n (1+k)/n \\ \vdots \end{Bmatrix} = -8\eta Ha \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{Bmatrix}. \tag{38}$$

For the case of a constant parameter, i.e. $\alpha = \alpha_0/2$, the above system can be resolved analytically and the only nonvanishing constant is recovered:

$$A_1 = \frac{-4k_1 aH}{\alpha_0 + 2(k_1 + k_2)a}. \tag{39}$$

Suppose now if $\alpha'_n = 0$, then system (31) reduces to

$$\begin{bmatrix} -\alpha_0 \mathbf{I} - 4\eta\Lambda^{-1} & \mathbf{C}^{\mathrm{t}}_\alpha \\ \mathbf{C}_\alpha & \alpha_0 \mathbf{I} - 4\eta\Lambda^{-1} \end{bmatrix} \begin{Bmatrix} \Lambda\mathbf{x} \\ \Lambda\mathbf{y} \end{Bmatrix} = 8\eta a H \begin{Bmatrix} \mathbf{h} \\ \mathbf{0} \end{Bmatrix}, \tag{40}$$

where $\mathbf{I}$ is an identity matrix.

For a numerical illustration, we consider the two sets of parameters $k_1 = 1, k_2 = 2,\ a = 1$, $\beta(\theta) = 3/2 + \cos\theta + \sin\theta$ for a weakly conducting interface and $k'_1 = 1,\ k'_2 = 1/2,\ a = 1,\ \alpha(\theta) = 1/(3/2 + \cos\theta + \sin\theta)$ for a highly conducting interface. For convenience, the corresponding solutions of the latter problem will be distinguished by a superscript prime. Evidently, the interface parameter $\beta(\theta)$, and also $\alpha(\theta)$, are always positive for all θ. Thus, the nonvanishing coefficients of $\beta(\theta)$ are $\beta_0 = 3, \beta'_1 = 1,\ \beta''_1 = 1$. Setting $\gamma_0 = 2,\ \gamma'_n = \gamma''_n = 0$ in (14), we find that for selecting $N = 15$ the numerical solutions for α'_n and α''_n converge to a sufficient accuracy. The Fourier coefficients of $\alpha(\theta)$ are calculated from Eq. (14) as

$$\begin{aligned} &\alpha'_{4n} = 4 \times \left(-\tfrac{1}{4}\right)^n, \quad \alpha'_{4n+1} = -2 \times \left(-\tfrac{1}{4}\right)^n, \quad \alpha'_{4n+2} = 0, \quad \alpha'_{4n+3} = \left(-\tfrac{1}{4}\right)^n, \\ &\alpha''_{4n+1} = -2 \times \left(-\tfrac{1}{4}\right)^n, \quad \alpha''_{4n+2} = 2 \times \left(-\tfrac{1}{4}\right)^n, \quad \alpha''_{4n+3} = -1 \times \left(-\tfrac{1}{4}\right)^n, \quad \alpha''_{4n+4} = 0, \end{aligned} \tag{41}$$

where $n = 0, 1, 2, \ldots$ which have been verified with direct expansions of the known formulae

$$\alpha'_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos n\theta}{3/2 + \cos\theta + \sin\theta}\,\mathrm{d}\theta, \qquad \alpha''_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin n\theta}{3/2 + \cos\theta + \sin\theta}\,\mathrm{d}\theta. \tag{42}$$

For a prescribed unit uniform intensity in the x_1 direction, the converged solutions for the weakly conducting interface are

$$\begin{aligned} &2A_0 = -0.67658, \quad x_1 = -1.1559, \quad x_2 = -0.12536, \quad x_3 = -9.4186 \times 10^{-3}, \\ &x_4 = 6.8896 \times 10^{-3}, \quad x_5 = -1.2086 \times 10^{-3}, \quad x_6 = 8.8964 \times 10^{-5}, \\ &x_7 = 3.7047 \times 10^{-6}, \quad x_8 = -1.6266 \times 10^{-6}, \quad x_9 = 1.8186 \times 10^{-7}, \\ &x_{10} = -8.8808 \times 10^{-9}, \end{aligned} \tag{43}$$

$$\begin{aligned} &y_1 = 0.17081, \quad y_2 = -0.18897, \quad y_3 = 4.6541 \times 10^{-2}, \quad y_4 = -4.5704 \times 10^{-3}, \\ &y_5 = -2.4458 \times 10^{-4}, \quad y_6 = 1.3411 \times 10^{-4}, \quad y_7 = -1.8306 \times 10^{-5}, \\ &y_8 = 1.079 \times 10^{-6}, \quad y_9 = 3.6803 \times 10^{-8}, \quad y_{10} = -1.3387 \times 10^{-8}. \end{aligned} \tag{44}$$

On the other hand, for a unit intensity applied in the x_2 direction, the solution for system (23) simply follows:

$$\begin{aligned} &A_0^\perp = A_0, \quad x^\perp_{4n+1} = y_{4n+1}, \quad x^\perp_{4n+2} = -x_{4n+2}, \quad x^\perp_{4n+3} = -y_{4n+3}, \quad x^\perp_{4n+4} = x_{4n+4}, \\ &y^\perp_{4n+1} = x_{4n+1}, \quad y^\perp_{4n+2} = y_{4n+2}, \quad y^\perp_{4n+3} = -x_{4n+3}, \quad y^\perp_{4n+4} = -y_{4n+4}, \end{aligned} \tag{45}$$

for $n = 1, 2, \ldots$ We remark that this recurrence relation (45) is indeed a consequence resulting from the symmetry of the considered function $\beta(\theta)$.

For the case with highly conducting interface, remarkably, the solutions for Eq. (31) are linked by those of the weakly conducting case by

$$\mathbf{x}' = \mathbf{y}^\perp, \quad \mathbf{y}' = -\mathbf{x}^\perp, \quad \mathbf{x}^{\perp\prime} = -\mathbf{y}, \quad \mathbf{y}^{\perp\prime} = \mathbf{x}. \tag{46}$$

Relation (46) agrees with the exact dual relations of two-dimensional composites with imperfect interfaces (Benveniste and Miloh, 1999).

5. Effective conductivity

Consider a two-phase composite medium consisting of equal-sized circular isotropic cylinder (with conductivity k_2) of radius a randomly dispersed in a homogeneous isotropic matrix of conductivity k_1. Homogeneous conditions of the temperature $T(S) = -H_i^0 x_i$ are prescribed on the boundary of the representative volume element. The effective conductivities k_{ij}^* are defined as

$$\langle q_i \rangle = k_{ij}^* \langle H_j \rangle, \tag{47}$$

where the bracket $\langle\ \rangle$ denotes the volume average over V.

For the β-type interface, the effective conductivity tensor $k_{ij}^{\beta *}$ can be expressed as (Benveniste and Miloh, 1986)

$$k_{ij}^{\beta *} H_j^0 = k_1 H_i^0 + c_2(k_2 - k_1)\langle H_i^{(2)} \rangle - k_1 \frac{c_2}{V_2} \int_\Gamma (T_2 - T_1) n_i \, \mathrm{d}S, \tag{48}$$

where $c_2 = V_2/V$ is the volume fraction of phase 2 and $\langle H_i^{(2)} \rangle$ denotes the average intensity $H_i^{(2)}$ of phase 2. For the α-type interface, it was shown that the effective conductivity tensor $k_{ij}^{\alpha *}$ can be written as (Miloh and Benveniste, 1999)

$$k_{ij}^{\alpha *} H_j^0 = k_1 H_i^0 + c_2(k_2 - k_1)\langle H_i^{(2)} \rangle - \frac{c_2}{V_2} \int_\Gamma (q_j^{(2)} n_j - q_j^{(1)} n_j) x_i \, \mathrm{d}S. \tag{49}$$

To provide an estimate for the effective conductivities (48) and (49) of the composite, we first consider a dilute approximation, which gives a reasonably accurate estimate for a small volume fraction of the inclusions (Christensen, 1979). In this method, the inclusions are far apart that all interactions between inclusions can be neglected. The averaged field quantities are then obtained by considering a solitary cylinder in an unbounded matrix and subjecting it at infinity to the homogeneous boundary conditions $T(S) = -H_i^0 x_i$.

For the case of the β-type interface, from the solution of Section 3.1, it is readily seen that

$$\langle H_1^{(2)} \rangle = \frac{-1}{V_2} \int_{V_2} \partial T_2 / \partial x_1 \, \mathrm{d}V = -A_1, \tag{50}$$

and, likewise

$$\langle H_2^{(2)} \rangle = -B_1. \tag{51}$$

Also,

$$\begin{aligned} &\frac{1}{V_2} \int_\Gamma (T_2 - T_1) n_1 \, \mathrm{d}S = (k+1) A_1 + 2H_1, \\ &\frac{1}{V_2} \int_\Gamma (T_2 - T_1) n_2 \, \mathrm{d}S = (k+1) B_1. \end{aligned} \tag{52}$$

Upon substituting Eqs. (50)–(52) into Eq. (48), we find

$$\begin{aligned} k_{11}^{\beta *} &= k_1 - c_2(k_2 - k_1) A_1/H_1 - c_2 k_1 [(k+1) A_1/H_1 + 2] = (1 - 2c_2) k_1 - 2c_2 k_2 A_1/H_1, \\ k_{21}^{\beta *} &= -c_2(k_2 - k_1) B_1/H_1 - c_2 k_1 (k+1) B_1/H_1 = -2c_2 k_2 B_1/H_1. \end{aligned} \tag{53}$$

To find $k_{12}^{\beta *}$ and $k_{22}^{\beta *}$, we apply a uniform intensity field H_2 in the x_2 direction. Analogous to Eq. (53), the effective properties follow:

$$k_{12}^{\beta*} = -c_2(k_2-k_1)A_1^{\perp}/H_2 - c_2k_1(k+1)A_1^{\perp}/H_2 = -2c_2k_2A_1^{\perp}/H_1,$$
$$k_{22}^{\beta*} = k_1 - c_2(k_2-k_1)B_1^{\perp}/H_2 - c_2k_1\left[(k+1)B_1^{\perp}/H_2+2\right] = (1-2c_2)k_1 - 2c_2k_2B_1^{\perp}/H_2. \tag{54}$$

It is noted that the overall properties solely depend on two coefficients A_1 and B_1. As seen from Eqs. (53) and (54), when $k_{11}^{\beta*} \neq k_{22}^{\beta*}$, the overall behavior of the composite becomes anisotropic. This phenomenon is due to the presence of a *variable* interface parameter. We mention that for the case of constant β $(=\beta_0/2)$, the effective conductivities of the composite is macroscopically isotropic. We have verified that this expression coincides with the previous known results.

For the case of the α-type interface, in analogy to Eqs. (50) and (51), we obtain

$$\langle H_1^{(2)}\rangle = -A_1', \qquad \langle H_2^{(2)}\rangle = -B_1', \tag{55}$$

and

$$\frac{1}{V_2}\int_{\Gamma}(q_j^{(2)}n_j - q_j^{(1)}n_j)x_1\,\mathrm{d}S = -k_1\left[(k+1)A_1' + 2H_1\right],$$
$$\frac{1}{V_2}\int_{\Gamma}(q_j^{(2)}n_j - q_j^{(1)}n_j)x_2\,\mathrm{d}S = -k_1(k+1)B_1'. \tag{56}$$

Upon substituting Eqs. (55) and (56) into Eq. (36), surprisingly, the effective conductivities $k_{11}^{\alpha*}$ and $k_{21}^{\alpha*}$ take the same forms as the right-hand side of Eq. (53)

$$k_{11}^{\alpha*} = k_1 - c_2(k_2-k_1)A_1'/H_1 + c_2k_1\left[(k+1)A_1'/H_1+2\right] = (1+2c_2)k_1 + 2c_2k_1A_1'/H_1,$$
$$k_{21}^{\alpha*} = -c_2(k_2-k_1)B_1'/H_1 + c_2k_1(k+1)B_1'/H_1 = 2c_2k_1B_1'/H_1. \tag{57}$$

Similarly, $k_{12}^{\alpha*}$ and $k_{22}^{\alpha*}$ take the forms

$$k_{12}^{\alpha*} = 2c_2k_1A_1^{\perp\prime}/H_2,$$
$$k_{22}^{\alpha*} = (1+2c_2)k_1 + 2c_2k_1B_1^{\perp\prime}/H_2. \tag{58}$$

For a constant α, our result can be exactly resolved which again recovers the previously known results (Miloh and Benveniste, 1999, Eq. (3.45)).

The estimate of the dilute approximation is only valid for a small volume concentration of the inclusions. To assess the effective conductivities when the concentration is nondilute, we employ the Mori–Tanaka mean field approach (Benveniste, 1987b). The framework of this model can be easily implemented by a slight modification from that of the dilute approximation. The formulation of the effective conductivities of a composite consisting of equal-size inclusions with weakly or highly conducting interfaces can be referred to the works of Benveniste (1987a) and Miloh and Benveniste (1999). For the present case, the effective conductivities can be written as

$$\mathbf{k}_{\mathrm{MT}}^{\beta*} = k_1\mathbf{I} + c_2(k_2-k_1)\mathbf{T}(c_1\mathbf{I}+c_2\mathbf{T}+c_2\mathbf{J})^{-1} - c_2k_1\mathbf{J}(c_1\mathbf{I}+c_2\mathbf{T}+c_2\mathbf{J})^{-1},$$
$$\mathbf{k}_{\mathrm{MT}}^{\alpha*} = k_1\mathbf{I} + c_2(k_2-k_1)\mathbf{M}(c_1\mathbf{I}+c_2\mathbf{M})^{-1} - c_2\mathbf{N}(c_1\mathbf{I}+c_2\mathbf{M})^{-1}, \tag{59}$$

where the matrices $\mathbf{T}$, $\mathbf{J}$, $\mathbf{M}$ and $\mathbf{N}$ are defined as

$$\mathbf{T} = -\begin{bmatrix} A_1/H_1 & A_1^{\perp}/H_2 \\ B_1/H_1 & B_1^{\perp}/H_2 \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} (1+k)A_1/H_1+2 & (1+k)A_1^{\perp}/H_2 \\ (1+k)B_1/H_1 & (1+k)B_1^{\perp}/H_2+2 \end{bmatrix},$$
$$\mathbf{M} = -\begin{bmatrix} A_1'/H_1 & A_1^{\perp\prime}/H_2 \\ B_1'/H_1 & B_1^{\perp\prime}/H_2 \end{bmatrix}, \qquad \mathbf{N} = -k_1\begin{bmatrix} (1+k)A_1'/H_1+2 & (1+k)A_1^{\perp\prime}/H_2 \\ (1+k)B_1'/H_1 & (1+k)B_1^{\perp\prime}/H_2+2 \end{bmatrix} \tag{60}$$

Table 1
Effective conductivities versus the volume fraction

c_2	$k_{1'}^{*\beta}/k_1$	$k_{2'}^{*\beta}/k_1$
0	1.00000	1.00000
0.1	0.93362	0.97716
0.2	0.87151	0.95483
0.3	0.81326	0.93300
0.4	0.75854	0.91166
0.5	0.70702	0.89078
0.6	0.65841	0.87035

in which the coefficients A_1, B_1, $A_1^{\perp}$, $B_1^{\perp}$ are the solutions of the auxiliary boundary-value problem (21) and (23), while the constants A_1', B_1', $A_1^{\perp\prime}, B_1^{\perp\prime}$ are found from Eqs. (31) and (33).

For a numerical illustration, we employ the two sets of parameters as in Section 4 to estimate the effective conductivity matrix. The effective conductivity matrix becomes anisotropic and its principal axes are along the directions of $x_{1'} = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $x_{2'} = (-1/\sqrt{2}, 1/\sqrt{2})$. The conductivities in the principal directions are listed in Table 1.

Interestingly, we observed from our calculations that the principal effective conductivity of the considered system are connected by

$$k_{2'}^{*\alpha} = 1/k_{1'}^{*\beta}, \qquad k_{1'}^{*\alpha} = 1/k_{2'}^{*\beta}. \tag{61}$$

For perfect bonding interfaces, this relation recovers the results of (Milton (1997)). In Section 6, we shall prove that connection (61) is indeed a general property for the effective conductivity tensor of the composite, without any regard to the detailed microgeometry, with variable interface parameters. Further, we showed that the field solutions of the boundary-value problem in Section 3 fulfill the reciprocal relations

$$B_1/H_1 = A_1^{\perp}/H_2. \tag{62}$$

This implies that the dilute estimate of the effective conductivity tensor is always symmetric. To examine whether or not the Mori–Tanaka estimate does provide a diagonally symmetric conductivity tensor, we restrict our attention to the considered system, namely a composite reinforced with circular inclusions with variable α or β. To start with, connection (62) implies that the matrices $\mathbf{T}$ and $\mathbf{M}$ in Eq. (60) are symmetric. From Eq. (60), we find that

$$\mathbf{J} = 2\mathbf{I} - (1+k)\mathbf{T}, \qquad \mathbf{N} = k_1(-2\mathbf{I} + (1+k)\mathbf{M}), \tag{63}$$

which suggests that $\mathbf{J}$, $\mathbf{N}$, $\mathbf{JT}^{-1}$, $\mathbf{NM}^{-1}$ and their inverse are all diagonally symmetric. Now, the Mori–Tanaka estimate of the effective conductivity tensor (74) can be recast as

$$\begin{aligned} \mathbf{k}_{\mathrm{MT}}^{\beta*} &= k_1\mathbf{I} + c_2(k_2 - k_1)\left(c_1\mathbf{T}^{-1} + c_2\mathbf{I} + c_2\mathbf{JT}^{-1}\right)^{-1} - c_2k_1\left(c_1\mathbf{J}^{-1} + c_2\mathbf{TJ}^{-1} + c_2\mathbf{I}\right)^{-1}, \\ \mathbf{k}_{\mathrm{MT}}^{\alpha*} &= k_1\mathbf{I} + c_2(k_2 - k_1)\left(c_1\mathbf{M}^{-1} + c_2\mathbf{I}\right)^{-1} - c_2\left(c_1\mathbf{N}^{-1} + c_2\mathbf{MN}^{-1}\right)^{-1}. \end{aligned} \tag{64}$$

Since each term in Eq. (64) is diagonally symmetric, $\mathbf{k}_{\mathrm{MT}}^{\beta*}$ and $\mathbf{k}_{\mathrm{MT}}^{\alpha*}$ are diagonally symmetric.

6. Microstructure independent relations

In this section, we present two microstructure independent properties regarding the composite system with variable interface parameters. The first one is on the diagonal symmetry of the effective conductivity tensor. The proof is achieved by extending first the well-known reciprocal theorem to the case of imperfect interfaces.

Proposition 1. *Consider the conduction problem of a composite body* (2*D or* 3*D*) *V* (*its outer boundary surface by S*), *in which the interfaces between the phases are of either weakly or highly conducting type Eqs.* (2) *and* (4) *characterized by an interface parameter that may not be constant along the interface. Let there be two states of equilibrium. One is given by*

$$T,\ q_i,\ H_i,$$

and the other by

$$T',\ q_i',\ H_i'.$$

It can be shown that the reciprocal relation holds

$$\int_S q_i' n_i T\,\mathrm{d}S = \int_S q_i n_i T'\,\mathrm{d}S. \tag{65}$$

Proof. Without loss of generality, it suffices to consider that the composite consists of two phases; each of the phase occupies region V_i $(V = V_1 + V_2)$, that are separated by an interface Γ. We first consider that the imperfect interface is of weakly conducting type. Using the divergence theorem together with Eq. (2) and $k_{ij} = k_{ji}$, the left-hand side of Eq. (65) is recast as

$$\begin{aligned} -\int_S q_i' n_i T\,\mathrm{d}S &= \int_{V_1} q_i^{1\prime} H_i^1\,\mathrm{d}V + \int_{V_2} q_i^{2\prime} H_i^2\,\mathrm{d}V + \int_\Gamma q_i^{1\prime} n_i (T_2 - T_1)\,\mathrm{d}S \\ &= \int_{V_1} q_i^1 H_i^{1\prime}\,\mathrm{d}V + \int_{V_2} q_i^2 H_i^{2\prime}\,\mathrm{d}V + \int_\Gamma \beta\big(T_2' - T_1'\big)(T_2 - T_1)\,\mathrm{d}S \\ &= \int_{V_1} q_i^1 H_i^{1\prime}\,\mathrm{d}V + \int_{V_2} q_i^2 H_i^{2\prime}\,\mathrm{d}V + \int_\Gamma q_i^1 n_i \big(T_2' - T_1'\big)\,\mathrm{d}S = -\int_S q_i n_i T'\,\mathrm{d}S. \end{aligned} \tag{66}$$

We thus prove that the reciprocal relation (65) is valid for a composite medium with a weakly conducting interface.

For the case of an interface with highly conducting type, one may proceed with the proof in the following steps, using the relation of Eq. (4a,b) and also $k_{ij} = k_{ji}$:

$$\begin{aligned} -\int_S q_i' n_i T\,\mathrm{d}S &= \int_{V_1} q_i^{1\prime} H_i^1\,\mathrm{d}V + \int_{V_2} q_i^{2\prime} H_i^2\,\mathrm{d}V + \int_\Gamma T_1\big(q_n^{2\prime} - q_n^{1\prime}\big)\,\mathrm{d}S \\ &= \int_{V_1} q_i^1 H_i^{1\prime}\,\mathrm{d}V + \int_{V_2} q_i^2 H_i^{2\prime}\,\mathrm{d}V + \int_\Gamma T_1 \nabla_{\mathrm{s}}\big(\alpha \nabla_{\mathrm{s}} T_1'\big)\,\mathrm{d}S \\ &= \int_{V_1} q_i^1 H_i^{1\prime}\,\mathrm{d}V + \int_{V_2} q_i^2 H_i^{2\prime}\,\mathrm{d}V + \int_\Gamma T_1' \nabla_{\mathrm{s}}(\alpha \nabla_{\mathrm{s}} T_1)\,\mathrm{d}S = -\int_S q_i n_i T'\,\mathrm{d}S. \end{aligned} \tag{67}$$

In proving the third equality of Eq. (67), we have employed the integral theorems of surface operators (Van Bladel, 1964, pp. 502–503). Thus, we have proved that Proposition 1 is valid for a solid with variably weakly or highly conducting interfaces. □

We remark that, although the reciprocal relation is well known for a solid with perfect interfaces in the contexts of elasticity and conductivity, to the author's knowledge, it has not been hitherto established in the case of imperfect interfaces of both kinds in conductivity problems. Proposition 1 allows us to show that for any variable interface parameter, the effective conductivity tensor is diagonally symmetric. To do this, let us consider the two fields in Proposition 2 be, respectively, subjected to the boundary conditions

$$T(S) = -H_1 x, \qquad T'(S) = -H_2' y. \tag{68}$$

The left-hand side of Eq. (65) can be rewritten as

$$-\int_S q_i' n_i T\,\mathrm{d}S = \int_S q_i' n_i H_1 x\,\mathrm{d}S = H_1 \int_V q_1'\,\mathrm{d}V = H_1 k_{12}^* H_2' V. \tag{69}$$

On the other hand, the right-hand side of Eq. (65) can be recast as

$$-\int_S q_i n_i T'\,\mathrm{d}S = \int_S q_i n_i H_2' y\,\mathrm{d}S = H_2' \int_V q_2\,\mathrm{d}V = H_2' k_{21}^* H_1 V. \tag{70}$$

Since the reciprocal theorem is valid for both types of interfaces, one readily proves that the effective conductivity tensor is always symmetric, i.e. $k_{12}^{\alpha *} = k_{21}^{\alpha *}$ and $k_{12}^{\beta *} = k_{21}^{\beta *}$. Similar steps can show that $k_{13}^{\alpha *} = k_{31}^{\alpha *}$, $k_{13}^{\beta *} = k_{31}^{\beta *}$ and $k_{23}^{\alpha *} = k_{32}^{\alpha *}$, $k_{23}^{\beta *} = k_{32}^{\beta *}$. We have now proven that the effective conductivity tensor with variable interface conditions is always diagonally symmetric.

Next, we prove a dual relation between the effective conductivity tensor of composites with variable interface parameters.

Proposition 2. *Consider a two-dimensional heterogeneous medium with variably imperfect interfaces. We will show that the effective conductivity matrices fulfill the connection*

$$\mathbf{k}^{*\beta}(k_1, k_2, \ldots, k_n, \beta(s)) = \mathbf{k}^{*\alpha}\left(\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_n}, \frac{1}{\beta(s)}\right) \Big/ \det \mathbf{k}^{*\alpha} \tag{71}$$

and that the effective conductivity of the composite in the principal directions, designated by $1'$ *and* $2'$, *follows the dual relations*

$$k_{2'}^{*\alpha}\left(\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_n}, \frac{1}{\beta(s)}\right) = 1/k_{1'}^{*\beta}(k_1, k_2, \ldots, k_n, \beta(s)), \tag{72}$$

$$k_{1'}^{*\alpha}\left(\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_n}, \frac{1}{\beta(s)}\right) = 1/k_{2'}^{*\beta}(k_1, k_2, \ldots, k_n, \beta(s)). \tag{73}$$

These results (61)–(73) also generalize the recent results of Lipton (1997) for composites with *constant* interface parameter to the systems with variable interface parameter. The proof of Proposition 2 is analogous to the spirit of the aforementioned works, and is thus only briefly outlined.

Proof. Following the concept of Milton (1997) for a two-dimensional composite medium with perfect interfaces, we define the dual fields (designated by a prime) $\mathbf{q}' = \mathbf{RH}$ and $\mathbf{H}' = \mathbf{Rq}$, where $\mathbf{R}$ is a rotation matrix for a 90° rotation about the x_3 axis so that the dual field $\mathbf{q}'$ is divergence free and $\mathbf{H}'$ is curl free. It then follows that

$$\mathbf{q}' = \mathbf{k}'(x_1, x_2)\mathbf{H}', \qquad \mathbf{k}'(x_1, x_2) = \mathbf{k}(x_1, x_2)/\det[\mathbf{k}(x_1, x_2)]. \tag{74a,b}$$

The boundary condition can be transformed into

$$\frac{\mathrm{d}T}{\mathrm{d}s} = -\mathbf{H}\cdot\mathbf{s} = -\mathbf{R}^{\mathrm{t}}\mathbf{q}'\cdot\mathbf{s} = -\mathbf{q}'\cdot\mathbf{n}, \tag{75}$$

where $\mathbf{n}$ is the outward normal to the boundary and $\mathbf{s} = -\mathbf{Rn}$ is the unit tangent vector along S. Thus a uniform intensity boundary condition in a given direction i is equivalent to a flux-type boundary condition perpendicular to i. The interface condition of a weakly conducting type (2) can be recast as

$$
\begin{array}{rrcl}
 & \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathbf{q}\cdot\mathbf{n}}{\beta} & = & \frac{\mathrm{d}}{\mathrm{d}s}(T_2 - T_1) \\
\Rightarrow & (\mathbf{q}\cdot\mathbf{n})\frac{-1}{\beta^2}\frac{\mathrm{d}\beta}{\mathrm{d}s} + \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{q}\cdot\mathbf{n}) & = & (\mathbf{H}_1 - \mathbf{H}_2)\cdot\mathbf{s} \\
\Rightarrow & (\mathbf{R}^{\mathrm{t}}\mathbf{H}'\cdot\mathbf{n})\frac{-1}{\beta^2}\frac{\mathrm{d}\beta}{\mathrm{d}s} + \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{R}^{\mathrm{t}}\mathbf{H}'\cdot\mathbf{n}) & = & \mathbf{R}^{\mathrm{t}}(\mathbf{q}_1' - \mathbf{q}_2')\cdot\mathbf{s} \\
\Rightarrow & (\mathbf{H}'\cdot\mathbf{s})\frac{1}{\beta^2}\frac{\mathrm{d}\beta}{\mathrm{d}s} - \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{H}'\cdot\mathbf{s}) & = & (\mathbf{q}_1' - \mathbf{q}_2')\cdot\mathbf{n} \\
\Rightarrow & \frac{-1}{\beta^2}\frac{\mathrm{d}\beta}{\mathrm{d}s}\frac{\mathrm{d}T'}{\mathrm{d}s} + \frac{1}{\beta}\frac{\mathrm{d}^2T'}{\mathrm{d}s^2} & = & (\mathbf{q}_1' - \mathbf{q}_2')\cdot\mathbf{n}.
\end{array} \tag{76}
$$

Thus, if one lets $\alpha(s) = 1/\beta(s)$, then Eq. (76) is exactly the highly conducting interface condition. We thus show that the fields of a composite medium with weakly conducting interface can be transformed into a dual medium with highly conducting interface in which the phase properties are given in Eq. (74a,b) and vice versa. By taking averages of Eq. (74b), it provides

$$
\mathbf{k}^{*\beta}(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \beta(s)) = \mathbf{k}^{*\alpha}\left(\mathbf{k}_1', \mathbf{k}_1', \ldots, \mathbf{k}_n', \frac{1}{\beta(s)}\right) \Big/ \det \mathbf{k}^{*\alpha}. \tag{77}
$$

Note that if $\mathbf{k}_i = k_i\mathbf{I}$, Eq. (77) exactly reduces to Eq. (71). Further, since $\det \mathbf{k}^{*\alpha}$ is a scalar, it suggests that $\mathbf{k}^{*\beta}$ and $\mathbf{k}^{*\alpha}$ possess the same principal directions. Let us write the principal conductivities of $\mathbf{k}^{*\alpha}$ and $\mathbf{k}^{*\beta}$ as

$$
\mathbf{k}^{*\alpha} = \mathrm{diag}\left[k_{1'}^{*\alpha}, k_{2'}^{*\alpha}\right], \qquad \mathbf{k}^{*\beta} = \mathrm{diag}\left[k_{1'}^{*\beta}, k_{2'}^{*\beta}\right]. \tag{78}
$$

Using the identity $\det \mathbf{k}^{*\alpha} = k_{1'}^{*\alpha} \times k_{2'}^{*\alpha}$, one can readily prove Eqs. (72) and (73). □

7. Closure

We have constructed a simple solution procedure for the considered boundary-value problem. Particularly, the solutions are governed by a linear set of infinite equations which can be readily resolved by an appropriate truncation. The governing matrix is primarily composed of elements which are simple combinations of the Fourier expansion coefficients of the interfacial parameter. We mention that for a different variation of β or α, the formulation remains unchanged simply by changing the Fourier coefficients of α or β. Although it may not be likely that the variations of the interface parameters could be actually measured or known a priori, the solutions may still provide useful information. For example, one may assess the influence of the variations of β or α on the degree of anisotropy of the effective medium. It appears also likely that in certain situations the introduction of a variable interface parameter around the inclusion-matrix interface may result in a fact that the matrix fields remain *undisturbed* upon the introduction of inclusion. The inclusion of this kind was referred to as "neutral inhomogeneity". Relevant studies have been explored in many different contexts (Mansfield, 1953; Benveniste and Miloh, 1999). We finally remark that the key feature of the formulation is governed by the fact that the admissible temperature field of a circle can be represented in a form akin to a Fourier series. Thus, by expanding the interface parameter by a trigonometric series, one may employ the orthogonality property to resolve the problem. The same thought can also be applied to an analogous problem for a spherical inclusion with variable interfacial parameter. In the latter case, the solutions are capable of being expressed in terms of expansions of spherical harmonics, in which an orthogonality relation is known to exist. Detailed results will be reported in a future publication.

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